APPLICATION OF A DIFFERENTIAL-DIFFERENCE METHOD TO THE SOLUTION OF ONE-DIMENSIONAL NONSTATIONARY HEAT CONDUCTION PROBLEMS WITH A MOVING BOUNDARY

L. Ya. Zhemoidina

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A differential-difference method is applied to obtain an approximate solution of one-dimensional nonstationary heat conduction problems with a moving boundary in rectangular and cylindrical systems of coordinates. Recursion formulas are obtained for the determination of successive values of the unknown functions.

Heat conduction problems with moving boundaries arise in processes in which the heat transfer is accompanied by a phase transformation in the conducting medium [1].

By way of example we cite the problems of melting and solidification of solids, the essential feature of which is the presence of a boundary separating phases of differing thermophysical properties, the boundary being displaced in time.

In a number of cases, for example, in the case of arcing spots in point-intensity arcs, critical sections of which are jets, etc., the thermal flows involved are so large that the material at the surface melts, is vaporized, and is carried away by the external flow. This problem is one of single phase. We consider several problems of this type.

Initially we consider the problem for the halfspace ($0 \le x < \infty$) under the assumption that its temperature stays the same at all points of each plane x = const, i.e., that it satisfies the one-dimensional heat conduction equation. Let us assume that a constant thermal flux q is present at the surface of the halfspace, and that initially the temperature throughout the halfspace is constant and taken to be equal to zero. The solution of the problem with a variable thermal flux, which depends on the time, may be obtained from the corresponding solutions for a constant thermal flux with the aid of Duhamel's theorem or by an application of the theorem involving the product of transforms [2].

The problem we have indicated here can, up to the onset of motion of the boundary, be formulated mathematically as follows: find the temperature distribution in the halfspace from the equation

$$c\rho \frac{\partial u(x,\ t)}{\partial t} = k \frac{\partial^2 u(x,\ t)}{\partial x^2} \tag{1}$$

in the domain $\Pi: \{0 < x < \infty,\ 0 < t < t_M\}$, with the additional conditions

$$u(x,\ t)|_{t=0} = 0, \quad u(x,\ t)|_{x=\infty} = 0, \quad -k \left.\frac{\partial u(x,\ t)}{\partial x}\right|_{x=0} = q. \tag{2}$$

The problem (1)-(2) is the usual problem of heat conduction with a fixed boundary, solvable with the aid of the Laplace transform. As is well known [2], this solution has the form

$$u(x,\ t) = \frac{q}{k}\left[2\sqrt{\frac{\varkappa t}{\pi}}\exp\left(-\frac{x^2}{4\varkappa t}\right) - x\frac{2}{\sqrt{\pi}}\int\limits_{\frac{x}{2\sqrt{\varkappa t}}}^{\infty}\exp(-z^2)\,dz\right], \tag{3}$$

where $\varkappa = k/c\rho$.

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We find the time, defining the start of the phase transition, and, consequently, the onset of motion of the boundary. Denoting the melting temperature by u_M, and the time needed to attain this temperature by t_M, we find from Eq. (3) for $x = 0$,

$$t_M = \frac{\pi k^2 u_M^2}{4\varkappa q^2}.$$

The temperature distribution at this time is as follows:

$$u(x, t)|_{t=t_M} = u_M \exp\left(-\frac{q^2x^2}{u_M^2 k^2 \pi}\right) - \frac{2qx}{k\sqrt{\pi}} \int\limits_{\frac{qx}{u_M k \sqrt{\pi}}}^{\infty} \exp(-z^2)\, dz. \tag{4}$$

We commence our calculations at the instant of melting. The problem may be formulated then as follows: determine the temperature of the halfspace, $u(x, t)$, and the position of its melting front, $x = \xi(t)$, from the equation

$$c\rho \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \tag{5}$$

in the domain $D: \{\xi(t) < x < \infty,\ 0 < t < T\}$, with the following initial and boundary conditions:

$$u(x, t)|_{t=0} = \varphi(x) = u_M \exp\left(-\frac{q^2x^2}{u_M^2 k^2 \pi}\right) - \frac{2qx}{k\sqrt{\pi}} \int\limits_{\frac{qx}{u_M k \sqrt{\pi}}}^{\infty} \exp(-z^2)\, dz, \tag{6}$$

$$0 < x < \infty, \quad \varphi(0) = u_M,$$

$$u(x, t)|_{x=\xi(t)} = u_M, \quad u(x, t)|_{x=\infty} = 0. \tag{7}$$

On the phase transition surface the following condition holds:

$$\lambda\rho \frac{d\xi(t)}{dt} = q + k \frac{\partial u(x, t)}{\partial x}\bigg|_{x=\xi(t)}, \quad 0 < t < T. \tag{8}$$

We assume that the unknown function $u(x, t)$ is defined and continuous in the domain $D^*: \{\xi(t) \le x < \infty,\ 0 \le t \le T\}$, that the derivatives $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are continuous and bounded in the interior of this domain, and that $\xi(t)$ is a continuous monotonically increasing positive twice-differentiable function for all $0 \le t \le T$, where $\xi(0) = 0$.

To solve this problem we apply a differential-difference method, i.e., a method involving time steps, an index denoting the number of the step.

In Eq. (5) we put $t = t_{m+1}$ and we replace the derivative with respect to t by the finite-difference ratio

$$\frac{\partial u(x, t)}{\partial t}\bigg|_{t=t_{m+1}} \approx \frac{u(x, t_{m+1}) - u(x, t_m)}{h}, \tag{9}$$

where $h = t_{m+1} - t_m$ denotes the time step.

Substituting the expression (9) into Eq. (5) and replacing the approximate equation by an exact one, we obtain a system of ordinary differential equations

$$\frac{d^2 u_{m+1}(x)}{dx^2} - a^2 u_{m+1}(x) = -a^2 u_m(x) \quad (m = 0, 1, 2, \ldots). \tag{10}$$

Here $a^2 = c\rho/kh$ and the $u_{m+1}(x)$ must approximate $u(x, t_{m+1})$. The initial and boundary conditions may be represented in the form

$$u(x, t)|_{t=0} = u_0(x) = \varphi(x), \quad 0 < x < \infty, \quad \varphi(0) = u_M,$$
$$u_{m+1}(x)|_{x=\xi(t_{m+1})} = u_M, \quad u_{m+1}(x)|_{x=\infty} = 0 \quad (m = 0, 1, 2, \ldots). \tag{11}$$

Similarly, if we replace the time-derivative $d\xi(t)/dt$ by the difference-ratio in Eq. (8), we obtain the following equations for determining the quantities ξ_{m+1}, which approximate the values $\xi(t_{m+1})$:

$$\xi_{m+1} = \xi_m + \frac{c}{a^2\lambda}\left[\frac{du_{m+1}(x)}{dx}\bigg|_{x=\xi_{m+1}} + \frac{q}{k}\right] \quad (m = 0, 1, 2, \ldots). \tag{12}$$

The solution of Eq. (10), obtained by the method of variation of parameters with the conditions (11) taken into account, and the fact that $u_{m+1}[\xi(t_{m+1})] \approx u_{m+1}(\xi_{m+1})$, may be represented in the form

$$u_{m+1}(x) = [u_M + a \int_{\xi_{m+1}}^{\infty} u_m(x_1) \operatorname{sh} a(x_1 - \xi_{m+1})\, dx_1] \exp[-a(x - \xi_{m+1})]$$

$$- a \int_{x}^{\infty} u_m(x_1) \operatorname{sh} a(x_1 - x)\, dx_1 \quad (m = 0,\ 1,\ 2,\ \ldots). \tag{13}$$

Expression (13) is a recursion relation for determining the successive values of $u_{m+1}(x)$.

Moreover convergence of the integrals is ensured by the behavior of the integrand function and Eq. (12) may be transformed as follows:

$$\xi_{m+1} = \xi_m + \frac{c}{\lambda}\left\{\int_{\xi_{m+1}}^{\infty} u_m(x_1) \exp[-a(x_1 - \xi_{m+1})]\, dx_1 + \frac{q}{a^2 k} - \frac{u_M}{a}\right\}. \tag{14}$$

We determine over each time step h the amount of displacement of the moving boundary.

We put

$$\xi_{m+1} - \xi_m = \delta_{m+1}.$$

Then

$$\xi_{m+1} = \xi_m + \delta_{m+1}. \tag{15}$$

Expression (15) serves as a recursion formula for finding successive values of ξ_{m+1}.

Let

$$F(\xi_{m+1}) = \int_{\xi_{m+1}}^{\infty} u_m(x_1) \exp[-a(x_1 - \xi_{m+1})]\, dx_1$$

then from Eq. (14) we obtain

$$\delta_{m+1} = \frac{c}{\lambda}\left[F(\xi_m + \delta_{m+1}) + \frac{q}{a^2 k} - \frac{u_M}{a}\right] \quad (m = 0,\ 1,\ 2,\ \ldots). \tag{16}$$

To find δ_{m+1} we expand the function $F(\xi_m + \delta_{m+1})$ by Taylor's formula

$$F(\xi_m + \delta_{m+1}) = F(\xi_m) + \delta_{m+1} F'(\xi_m) + O(\delta_{m+1}^2).$$

Taking note of terms of order δ_{m+1}^2 and the equality $F'(\xi_m) = aF(\xi_m) - u_M$, we obtain from Eq. (16)

$$\delta_{m+1} \approx \frac{F(\xi_m) + \dfrac{q}{a^2 k} - \dfrac{u_M}{a}}{\dfrac{\lambda}{c} - aF(\xi_m) + u_M} \quad (m = 0,\ 1,\ 2,\ \ldots). \tag{17}$$

The values of δ_{m+1}, found from formula (17), permit us to successively determine ξ_{m+1} for m = 0, 1, 2, . . . , since $\xi_0 = 0$ is known.

A problem, similar to the preceding, for the case of a layer of finite thickness ($0 \le x \le l$), where in the melting process the temperature on its lower surface (x = 0) is a function of the time, f(t), and where $u_M = 0$, yields the following solutions:

$$u_{m+1}(x) = f_{m+1} \frac{\operatorname{sh} a(\xi_{m+1} - x)}{\operatorname{sh} a\,\xi_{m+1}} + a \frac{\operatorname{sh} ax}{\operatorname{sh} a\xi_{m+1}} \int_0^{\xi_{m+1}} u_m(x_1) \operatorname{sh} a\,(\xi_{m+1} - x_1)\, dx_1 - a \int_0^x u_m(x_1) \operatorname{sh} a(x - x_1)\, dx_1$$

$$(m = 0,\ 1,\ 2,\ \ldots),$$

$$\delta_{m+1} \approx \frac{\Phi(\xi_m) + \dfrac{q_{m+1}}{a^2 k}}{\dfrac{\lambda}{c} - a \operatorname{cth} a\xi_m \Phi(\xi_m)},$$

where

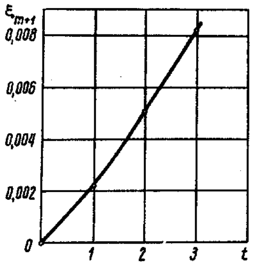

Fig. 1. Curve showing the dependence of the thickness of the melted portion of the wall (in meters) on the time (in secs).

TABLE 1

m	0	1	2
δ_{m+1}	0,223	0,279	0,321
ξ_{m+1}	0,223	0,502	0,823

$$\Phi(\xi_m) = -\frac{1}{\operatorname{sh} a\xi_m}\left(\frac{f_{m+1}}{a} + \int_0^{\xi_m} u_m(x_1)\operatorname{sh} ax_1 dx_1\right)$$

and $\xi_{m+1} = \xi_m - \delta_{m+1}$ for m = 0, 1, 2, . . . , where $\xi_0 = l$.

If we also consider the problems for the cases of continuous and hollow cylinders, also for zero melting temperature, we obtain the solutions:

a) for the continuous cylinder

$$u_{m+1}(r) = a^2\int_0^r r_1 u_m(r_1)\tilde{k}(r, r_1)\,dr_1 - \frac{a^2 I_0(ar)}{I_0(a\eta_{m+1})}\int_0^{\eta_{m+1}} r_1 u_m(r_1)\tilde{k}(\eta_{m+1}, r_1)\,dr_1$$

$$(m = 0, 1, 2, \ldots),$$

$$\delta_{m+1} \approx \frac{Q(\eta_m) + \frac{q_{m+1}}{a^2 k}}{\frac{\lambda}{c} + Q'(\eta_m)},$$

where

$$Q(\eta_m) = -\frac{1}{\eta_m I_0(a\eta_m)}\int_0^{\eta_m} r_1 u_m(r_1) I_0(ar_1)\,dr_1;$$

b) for the hollow cylinder

$$u_{m+1}(r) = f_{m+1}\frac{\tilde{k}(r, \eta_{m+1})}{\tilde{k}(r_0, \eta_{m+1})} + a^2\frac{\tilde{k}(r, r_0)}{\tilde{k}(r_0, \eta_{m+1})}\int_{r_0}^{\eta_{m+1}} r_1 u_m(r_1)\tilde{k}(\eta_{m+1}, r_1)\,dr_1$$

$$+ a^2\int_{r_0}^{r} r_1 u_m(r_1)\tilde{k}(r, r_1)\,dr_1 \quad (m = 0, 1, 2, \ldots),$$

$$\delta_{m+1} \approx \frac{S(\eta_m) + \frac{q}{a^2 k}}{\frac{\lambda}{c} + S'(\eta_m)},$$

where

$$S(\eta_m) = -\frac{1}{\eta_m \tilde{k}(r_0, \eta_m)}\left[\frac{f_{m+1}}{a^2} + \int_{r_0}^{\eta_{m+1}} r_1 u_m(r_1)\tilde{k}(r_0, r_1)\,dr_1\right].$$

In both cases $\eta_{m+1} = \eta_m - \delta_{m+1}$, $\eta_0 = R$, and $I_0(ar)$, $K_0(ar)$ are Bessel functions of zero order of the first and second kind of an imaginary argument [4],

$$\tilde{k}(r,\ r_1) = I_0(ar_1)\,K_0(ar) - I_0(ar)\,K_0(ar_1).$$

By way of example we give below the results of the calculations for the first few instants in the instantaneous melting of a steel wall. The thermophysical parameters of steel are as follows: $c = 5 \cdot 10^2$ J /kg·deg, $\rho = 7.8 \cdot 10^3$ kg/m^3, $\lambda = 1.38 \cdot 10^5$ J/kg; $u_M = 1803$°K; k = 46 βT/m·deg. The calculations were made with a constant heat flux $q = 2.09 \cdot 10^7$ βT/m^2. In addition, h = 1 sec, $a = 2.92 \cdot 10^2$ 1/m. The melting curve of ξ_{m+1} is shown in the figure.

An estimate of the error was obtained using Rothe's method [5-7]. The errors made were shown to be of order O(h).

NOTATION

u(x, t), u(r, t)	are the temperature;
ξ(t)	is the function determining the position of moving boundary;
η(t)	is the variable radius of cylinder;
c	is the specific heat;
ρ	is the density;
k	is the thermal conductivity;
$\varkappa$	is the thermal diffusivity;
λ	is the latent heat of fusion;
T	is the time of process duration under consideration.

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